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Abstract

This paper develops a model of competitive bidding with a resale market. The primary market is modelled as a common-value auction, in which bidders participate for the purpose of resale. After the auction the winning bidders sell the objects in a secondary market and the buyers on the secondary market receive information about the bids submitted in the auction. The effect of this information linkage between the primary auction and the secondary market on bidding behaviour in the primary auction is examined. The auctioneer's expected revenues from organizing the primary market as a discriminatory auction versus a uniform-price auction are compared, and plausible sufficient conditions under which the uniform-price auction yields higher expected revenues are obtained. An example of our model, with the primary market organized as a discriminatory auction, is the U.S. Treasury bill market.

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1 Introduction

One of the major achievements of economic theory in the past decade has been a deeper understanding of the conduct and design of auctions. Recent surveys of the literature on auctions are McAfee and McMillan (1987), Milgrom (1987), and Wilson (1988). Auctions account for a large volume of economic and financial activities. The U.S. Interior Department uses a sealed-bid auction to sell mineral rights on federally owned properties. Auction houses regularly conduct auctions of antiques, jewelry, and works of art. Every week the U.S. Treasury Department uses a sealed-bid auction to sell Treasury bills worth billions of dollars. Many other economic and financial transactions, although not explicitly conducted as auctions, can nevertheless be thought of as implicitly carried out through auctions; see, for example, an analysis of sales of seasoned new issues in Parsons and Raviv (1985), and the market for corporate control in Tiemann (1986).

One feature often shared by such financial activities is that there exist active resale or secondary markets for the objects for sale. This is true, for example, for Treasury bills and for seasoned new issues. One may argue that when there exists the possibility of resale, the auction will be common-value with the resale price being the common valuation among all the participating bidders.[1] Thus it might be argued that the theory of common-value auctions developed by Wilson (1969), and Milgrom and Weber (1982a) is applicable. The observation that an auction with a resale market is common-value is certainly true. However, there are situations that make existing theory inapplicable. A case in point is Treasury bill auctions.

In Treasury bill auctions, there are usually about forty bidders or ***primary dealers*** who participate in the weekly auction. These primary dealers are large financial institutions. They submit ***competitive*** sealed bids that are price-quantity pairs. Others, usually individual investors, can submit ***noncompetitive*** sealed bids that specify quantity (less then a prespecified maximum). The noncompetitive bids, usually small in quantity, always win. The primary dealers compete for the remaining bills in a discriminatory auction. That is, the demands of the bidders, starting with the highest price bidder down, are met until all the bills are allocated. The winning competitive bidders pay the unit price they submitted. All the noncompetitive bidders pay the quantity-weighted average price of all the winning competitive bids. After the auction, the Treasury department announces some summary

[1] An auction is common-value if participating bidders do not value the objects for sale differently.

statistics about the bids submitted. These include

- total tender amount received;
- total tender amount accepted;
- highest winning bid;
- lowest winning bid;
- quantity weighted average of winning bids; and
- the split between competitive and noncompetitive bids.

The Treasury bills are then delivered to the winning bidders and can be resold at an active secondary market.

Since primary bidders are large institutions, they tend to have private information about the term structure of interest rates that is better than the information possessed by investors in the secondary markets. The primary dealers submit bids in the auction based on both information that is publicly available at the time, and their private information. The buyers on the resale market have access only to public information, including information revealed by the Treasury about the bids submitted in the auction. To the extent that bids submitted reveal the private information of the primary dealers, the resale price in the secondary market will be responsive to the bids. This creates an incentive for the primary dealers to *signal* their private information to the secondary market participants. This information linkage between the actions taken by the bidders in the auction and the resale price is absent in existing models of common-value auctions and is the primary focus of this paper.[2]

In Section 2, we develop a model of competitive bidding with a resale market. The primary dealers or bidders are risk-neutral and have private information about the true value of the objects. We assume that the bidders' private signals and the true value are *affiliated* random variables, i.e., roughly speaking, higher realizations of a bidder's private signal imply that higher realizations of the true value, and of the other bidders' private signals, are more likely. After the auction the auctioneer publicly announces some information (the prices paid by the winning bidders, for example) about the auction. The winning bidders

[2]For an analysis of bidding with a resale market when the valuations of the bidders are common knowledge, see Milgrom (1987).

then sell the objects in the secondary market, at a price equal to the expected value of the object conditional on all public information.[3]

Although the primary motivation of this model is the Treasury bill market, there are many institutional details, which are absent. For instance, we require that competitive bidders demand at most one unit of the object, instead of being allowed to choose quantity. And we do not model the effect of any forward contracts, and close substitutes (such as last week's Treasury bills) owned by primary bidders on their bidding strategies. In this paper we focus on one aspect of the Treasury bill market — the informational linkage of the resale market and the primary auction.

The results of this paper may also be helpful in analyzing other types of auctions with resale markets, such as art auctions, in which bidders have correlated values. If a painting by Van Gogh is auctioned at a price much higher than expected, then one might expect this and other paintings by Van Gogh to be sold at higher prices in future.

In section 3 we analyze discriminatory auctions. It is assumed that the winning bids and the highest losing bids are revealed at the end of the auction. We provide sufficient conditions for the existence of a unique symmetric Nash equilibrium in nondecreasing strategies in the auction. Unlike the model in Milgrom and Weber (1982a) where bidders participate for the purpose of consumption and there is no resale market, the affiliation property alone is not sufficient for the existence of an equilibrium. The equilibrium bids we obtain are higher than those derived in Milgrom and Weber (1982a), because primary bidders have an incentive to *signal*.

A key insight gained from the theory of common-valuation auctions without resale markets is that the greater the amount of information (about the true value of the objects for sale) revealed in an auction, the greater the expected revenue for the auctioneer. Any reduction in the uncertainty about the true value weakens the *winners' curse* which in turns causes the bidders to bid more aggressively. Thus if the auctioneer has private information affiliated with the bidders' valuations, he can increase his expected revenue by precommitting to announcing his private information before the auction. When there is a resale market and there exists an incentive for the bidders to signal, the auctioneer may reduce the bidders' incentive to signal and thus decrease his expected revenue if he announces his

[3]Riley (1988) investigates a model in which conditional upon winning, each bidder's payment depends upon all the bids submitted. In our model, the expected value for each primary bidder depends on the bids submitted.

private information. Sufficient conditions for the public announcement of the auctioneer's private information to increase his expected revenue are provided.

In Sections 4 and 5 we consider a uniform-price auction, that is an auction in which the rule for determining the winning bidders is identical to the one in the discriminatory auction, but the winning bidders pay a uniform price equal to the highest losing bid. The existence of an equilibrium depends in part on the kind of information about the auction publicly revealed by the auctioneer. If, as we assumed for discriminatory auctions, the winning bids are announced then we show by example that there may exist an incentive for the bidders to submit arbitrarily large bids in order to deceive the secondary market buyers. Bidders in a discriminatory auction do not have such an incentive since, upon winning, they must pay what they bid. However, a symmetric Nash equilibrium always exists in the uniform-price auction, provided that only the price paid by the winning bidders is announced (and thus bidders have no incentive to signal).

Next, we turn to the question of the auctioneer's revenues. The key insight gained from the theory of auctions without resale markets mentioned above is also useful here. Without resale markets, uniform-price auctions yield higher revenues than discriminatory auctions, since in the former the price is linked to the information of the highest losing bidder. When there are resale markets in which the buyers draw inferences about the true value from the bids and the winning bids are announced, there is an additional factor which works in the same direction. Namely that in a uniform-price auction it is cheaper to bid high in order to signal a high realization of private information, since conditional upon winning a bidder does not pay what he bids. Therefore, in our model as well, uniform-price auctions result in greater expected revenues for the auctioneer, when there exists an equilibrium.

We also provide plausible sufficient conditions under which the auctioneer's revenues when the primary auction is organized as a uniform-price auction and only the price paid by winning bidders is announced is greater than under a discriminatory auction when the winning bids *are* announced. Section 6 contains concluding remarks. All proofs are in an appendix.

The revenue maximizing mechanism for selling Treasury bills was a subject of debate in the early 1960s. Friedman (1960) proposed that the Treasury should switch from a discriminatory auction to a uniform-price auction for the sale of Treasury bills. Apart from the fact that uniform-price auctions would induce bidders to reveal their true demand curves, Friedman asserted that discriminatory auctions encouraged collusion and discouraged smaller

bidders from participating. Both Goldsein (1960) and Brimmer (1962) disputed Friedman's contention. Smith (1966), on the basis of a mathematical model, concluded that uniform-price auctions yield greater revenues. Unlike Smith's model, our model is game-theoretic in that each bidder's beliefs about the others' bids are confirmed in an equilibrium, and we model the information linkage between the primary auction and the secondary market. Like Smith, our analysis provides support for Friedman's proposal that the Treasury bill auction should be uniform-price.

2 The model

Consider a common-value auction in which n risk-neutral bidders (the dealers who submit competitive bids in Treasury bill auctions) bid for k identical, indivisible objects, with $n > k$. The true value of the objects is the same for all the bidders, and is unknown to them at the time they submit bids. Each bidder privately observes a signal about the true value, based on which he submits a bid. We assume that there are no noncompetitive bids. In Section 6 we indicate how noncompetitive bids can be incorporated in our model. Throughout we assume that each bidder demands (or is allowed) at most one unit of the object.

The primary dealers' interest in the objects being auctioned is solely for the purpose of resale in the secondary market. We assume that the primary dealers' personal (consumption) valuations of the object are always sufficiently lower than the valuations of the resale market buyers that they would prefer to resell the objects rather then consume them. For instance, the primary bidders may have capital constraints so that unless they sell this week's Treasury bills, they may not be able to participate in next week's auction. Since, as we will establish, primary bidders make positive expected profits in the auction, they might prefer to sell the Treasury bills at their expected value conditional on all publicly known information.

If instead one assumes that the primary bidders' personal value of the object is at the same level as that of the resale buyers, then if all the private information of the winning primary bidders is not revealed after the primary auction, the primary bidders will never sell the object if the expected value of the object conditional on all public information and their privately known information is greater than the resale price, and will always sell when their expected value is strictly less than the resale price. Therefore the resale market buyers will make strictly negative expected profits and the resale market will break down.

Hence we preclude this possibility. For similar reasons we assume that primary bidders can participate in the resale market only as sellers, not as buyers.

After the auction the auctioneer publicly announces some information about the auction. For simplicity we assume that that the winning bids and the highest losing bid submitted in the auction are publicly announced. In the case of the uniform-price auction, there may not exist an equilibrium if the winning bids are revealed at the end of the auction. Therefore, we also analyze uniform-price auctions when the winning bids are not announced and only the price paid by the winning bidders is announced.

We allow the possibility that some additional information about the value of the objects for sale may become publicly available after the bids are submitted, but before the opening of the secondary market. The k winners in the primary auction then sell the objects to risk-neutral buyers on the secondary markets. The buyers on the secondary markets do not have access to any private information about the true value. They infer what they can from the information released by the auctioneer about the primary auction, and any other publicly available information. Thus, regardless of the secondary market mechanism — an auction or a posted price market — the resale price will be the expected value of the object conditional on all publicly available information.[4]

The n risk-neutral bidders will be indexed by $i = 1, 2, ..., n$. The true value of each object being auctioned is a random variable, $\tilde{V}$. Each bidder i has a common prior on $\tilde{V}$, and observes a private signal, $\tilde{X}_i$, about the true value. Let $\tilde{P}$ denote any other information that becomes public after the auction is over but before the resale market meets. We will assume, except when otherwise stated, that given $\tilde{P}$ the bidders' signals are not uninformative about the true value, that is, $\mathbf{E}[\tilde{V}|\tilde{P}] \not\equiv \mathbf{E}[\tilde{V}|\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n, \tilde{P}]$. If this condition is violated, for example when $\tilde{P} \equiv \tilde{V}$, our model reduces to the usual common-value auction without a resale market.

Let $f(v, p, \mathbf{x})$ denote the joint density function of $\tilde{V}$, $\tilde{P}$, and the vector of signals $\tilde{X} \equiv (\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)$. It is assumed that f is symmetric in the last n arguments. Let $[\underline{v}, \overline{v}] \times [\underline{p}, \overline{p}] \times [\underline{x}, \overline{x}]^n$ be the support of f, where $[\underline{x}, \overline{x}]^n$ denotes the n-fold product of $[\underline{x}, \overline{x}]$. Note that we do not rule out the possibility that the support of the random variables is unbounded either from above or from below. Further, it is assumed that all the random variables in

[4]The participants in the secondary market do not have to be risk neutral. In the case of Treasury bill auctions, the "true" value of a bill for a primary bidder that pays say \$1 in 182 days will be $\mathbf{E}[\tilde{m}|\mathcal{F}]$, where $\tilde{m}$ denotes the random marginal rate of substitution between consumption 182 days from now and that of today and where $\mathcal{F}$ denotes the information revealed by all the bids submitted. This true value will be the secondary market price whether or not the secondary market participants are risk neutral.

this model are *affiliated*. That is, for all $\mathbf{x}, \mathbf{x}' \in [\underline{x}, \overline{x}]^n$, v, $v' \in [\underline{v}, \overline{v}]$, and p, $p' \in [\underline{p}, \overline{p}]$,

$$f((v,p,\mathbf{x}) \vee (v',p',\mathbf{x}'))f((v,p,\mathbf{x}) \wedge (v',p',\mathbf{x}')) \geq f(v,p,\mathbf{x})f(v',p',\mathbf{x}'),$$

where $\vee$ denotes the componentwise maximum, and $\wedge$ denotes the componentwise minimum. Affiliation is said to be strict if the above inequality is strict. Affiliation implies that if H is an increasing[5] function then $\mathbf{E}[H(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)|c_i \leq \tilde{X}_i \leq d_i,\ i = 1, \ldots, n]$ is an increasing function of c_i, d_i. The reader is referred to Milgrom and Weber (1982a) for other implications of affiliation. We further assume for simplicity that if H is continuously differentiable then $\mathbf{E}[H(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)|c_i \leq \tilde{X}_i \leq d_i,\ i \leq n]$ is continuously differentiable in c_i and d_i, for all $c_i, d_i \in [\underline{x}, \overline{x}]$, with the convention that the derivative at $\underline{x}$ is the right-hand derivative and at $\overline{x}$ is the left-hand derivative. Moreover, we shall assume that $(\tilde{V}, \tilde{P}, \tilde{X}_1, \ldots, \tilde{X}_n)$ are strictly affiliated so that if H is strictly increasing in any of $(\tilde{V}, \tilde{P}, \tilde{X}_1, \ldots, \tilde{X}_n)$, say in $\tilde{X}_1$, then $\mathbf{E}[H(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)|c_i \leq \tilde{X}_i \leq d_i,\ i \neq 1]$ is strictly increasing in c_i and d_i for all $c_i, d_i \in [\underline{x}, \overline{x})$.

3 Discriminatory auction

In a discriminatory auction, the bidders submit sealed bids and the k highest bidders win the auction. A winning bidder pays the price that he or she bids. In this section we show that when the bidders' private signals are *information complements*[6], in a sense to be defined later, there exists a symmetric Nash equilibrium with strictly increasing strategies in the bidding game among the primary dealers. Unlike the auctions examined in Milgrom and Weber (1982a), the affiliation property alone is not sufficient for the existence of a Nash equilibrium. Intuitively, when the motive of the primary bidders is to resell in a secondary market in which the buyers know some or all of their bids (or a summary statistic based on their bids), there exists an incentive for the primary bidders to bid more than they otherwise would and thus *signal* their private information. This is because, by affiliation, the resale value is responsive to the bids submitted to the extent the bids reveal the private information received by the primary bidders. If each bidder's incentive to signal increases with his information realization, then there exists an equilibrium in strictly increasing strategies. It is the information complementarity of the bidders' signals with respect to the true value that

[5]Throughout this paper, we will use weak relations. For example, increasing means nondecreasing, positive means nonnegative, etc. If a relation is strict, we will say, for example, strictly increasing.

[6]The reader will see that our notion of information complementarity is different from that in Milgrom and Weber (1982b)

ensures that the bidders' incentive to signal increases with their information realizations and enables them to sort themselves in a separating equilibrium. We also show that if secondary markets participant's beliefs are *monotone*, in a sense to be defined, then there exists a unique symmetric equilibrium.

In a model where bidders participate in an auction for final consumption of the objects, Milgrom and Weber (1982a) show that the auctioneer's expected revenue can be increased if he precommits to truthfully reporting his private information about the objects for sale before the auction, provided that his private information is affiliated with the bidders' private information. This follows since by publicly announcing his information, the auctioneer introduces an additional source of affiliation among the primary bidders' private information and thus weakens the winners' curse. Hence the bidders compete more aggressively and the expected selling price is increased. However, in our model with a resale market this result is not necessarily true. A portion of the bid submitted by a bidder is attributed to his incentive to signal to the resale market participants. If the auctioneer's private information is a "substitute" for the bidders' information, announcing that information will reduce the responsiveness of the resale price to the bidders' information. This in turn reduces the incentive for the bidders to signal and may cause the expected selling price to fall. On the other hand, when the auctioneer's private information is a "complement" to that of the bidders', it is always beneficial for the auctioneer to announce his private information.

3.1 Existence of a symmetric Nash equilibrium

By the hypothesis that $f(v,p,\mathbf{x})$ is symmetric in its last n arguments, this is a symmetric game. Thus it is natural to investigate the existence of a symmetric Nash equilibrium. We examine the game from bidder 1's point of view. The analysis from the other bidders' viewpoint is symmetric.[7] Note that at the time when bidder 1 submits his bid, he only observes his private information $\tilde{X}_1$. Thus a strategy for bidder 1 is a function of $\tilde{X}_1$. Bidder $i's$ strategy is denoted $b_i : [\underline{x}, \overline{x}] \rightarrow \Re$. We begin our analysis by deriving the first-order necessary conditions for an n-tuple $(\hat{b}, \ldots, \hat{b})$ to be a Nash equilibrium in strictly increasing and differentiable strategies, when buyers in the secondary market believe that $(\hat{b}, \ldots, \hat{b})$ are the strategies followed in the bidding.

[7] To simplify the analysis we arbitrarily assume throughout that in case of a tie the winner is not chosen randomly. Rather, bidder 1 is declared the winner. This assumption is inconsequential. The equilibrium strategy will remain unchanged if we assume that in case of a tie, the winner(s) is (are) chosen from the tied bidders at random.

Since buyers in the secondary market do not have access to private information about the true value, the resale price is the expectation of $\tilde{V}$ conditional on all public information. As mentioned earlier, to simplify the analysis we assume that the auctioneer announces the prices paid by winning bidders (i.e., the winning bids) and the highest losing bid.[8] Suppose that bidders $i = 2, \ldots, n$ adopt the strategy $\hat{b}$, bidder 1 receives information $\tilde{X}_1 = x$ and submits a bid equal to b. Then if bidder 1 wins with a bid b the resale price will be

$$\begin{aligned} r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) &\equiv \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 = \hat{b}^{-1}(b), \hat{b}^{-1}(\hat{b}(\tilde{Y}_1)), \ldots, \hat{b}^{-1}(\hat{b}(\tilde{Y}_k), \tilde{P}\right] \\ &= \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 = \hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}\right], \end{aligned} \tag{1}$$

where $\hat{b}^{-1}$ denotes the inverse[9] of $\hat{b}$ and $\tilde{Y}_j$ is the j-th order statistic of $(\tilde{X}_2, \ldots, \tilde{X}_n)$. Note that if $\tilde{P} \equiv \tilde{V}$, $r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) \equiv \tilde{V}$, and our model reduces to an ordinary common-value auction without a resale market. Define

$$v^d(x', x, y) \equiv \mathbf{E}\left[r^d(x', \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) \middle| \tilde{X}_1 = x, \tilde{Y}_k = y\right]. \tag{2}$$

By our hypothesis about strict affiliation, both r^d and v^d are strictly increasing in each of their arguments (provided that given $\tilde{P}$, the bidders' signals are not uninformative about $\tilde{V}$). The expected profit for bidder 1 when $\tilde{X}_1 = x$ and he submits a bid equal to b is

$$\begin{aligned} \Pi^d(b|x) &\equiv \mathbf{E}\left[\left(r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b\right) 1_{\{b \geq \hat{b}(\tilde{Y}_k)\}} \middle| X_1 = x\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\left(r^d(\hat{b}^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b\right) 1_{\{b \geq \hat{b}(\tilde{Y}_k)\}} \middle| \tilde{X}_1, \tilde{Y}_k\right] \middle| \tilde{X}_1 = x\right] \\ &= \mathbf{E}\left[\left(v^d(\hat{b}^{-1}(b), \tilde{X}_1, \tilde{Y}_k) - b\right) 1_{\{b \geq \hat{b}(\tilde{Y}_k)\}} \middle| \tilde{X}_1 = x\right], \\ &= \int_{\underline{x}}^{\hat{b}^{-1}(b)} (v^d(\hat{b}^{-1}(b), x, y) - b) f_k(y|x) dy, \end{aligned}$$

where the second equality follows from the law of iterative expectations, and $f_k(y|x)$ denotes the conditional density function of $\tilde{Y}_k$ given $\tilde{X}_1$. Taking the first derivative of $\Pi^d(b|x)$ with respect to b gives

$$\begin{aligned} \frac{\partial \Pi^d(b|x)}{\partial b} = &\left(v^d(\hat{b}^{-1}(b), x, \hat{b}^{-1}(b)) - b\right) f_k(\hat{b}^{-1}(b)|x)(\hat{b}'(\hat{b}^{-1}(b)))^{-1} - F_k(\hat{b}^{-1}(b)|x) \\ &+ (\hat{b}'(\hat{b}^{-1}(b)))^{-1} \int_{\underline{x}}^{\hat{b}^{-1}(b)} v_1^d(\hat{b}^{-1}(b), x, y) f_k(y|x) dy, \end{aligned} \tag{3}$$

where $\hat{b}'(x)$ is the derivative of $\hat{b}(x)$, $F_k(y|x)$ is the conditional distribution function of $\tilde{Y}_k$ given $\tilde{X}_1$, and v_1^d is the partial derivative of v^d with respect to its first argument. For

[8] If some of the other losing bids, or some function of them, are also announced all the results remain unchanged.

[9] If $b < \hat{b}(\underline{x})$ then $\hat{b}^{-1}(b) = \underline{x}$ and if $b > \hat{b}(\overline{x})$ then $\hat{b}^{-1}(b) = \overline{x}$. Thus we only need to consider values of b that lie in the range of $\hat{b}$.

$(\hat{b},\ldots,\hat{b})$ to be a Nash equilibrium, it is necessary that $\hat{b}$ be a best response for bidder 1 when bidders $i = 2,\ldots,n$ adopt strategy $\hat{b}$ and the resale market participants believe that all bidders adopt $\hat{b}$. That is, relation (3) must be zero when $b = \hat{b}(x)$:

$$0 = \left.\frac{\partial\Pi^d(b|x)}{\partial b}\right|_{b=\hat{b}(x)} \begin{array}{l} = \left(v^d(x,x,x) - \hat{b}(x)\right) f_k(x|x)(\hat{b}'(x))^{-1} - F_k(x|x) \\ \quad +(\hat{b}'(x))^{-1}\int_{\underline{x}}^{x} v_1^d(x,x,y) f_k(y|x)dy. \end{array} \tag{4}$$

Rearranging (4) gives an ordinary differential equation:

$$\hat{b}'(x) = (v^d(x,x,x) - b(x))\frac{f_k(x|x)}{F_k(x|x)} + \int_{\underline{x}}^{x} v_1^d(x,x,y)\frac{f_k(y|x)}{F_k(x|x)}dy. \tag{5}$$

Note that, by the definition of v^d and the law of iterative expectations,

$$v^d(x,x,y) = \mathbf{E}\left[\tilde{V}\middle|\tilde{X}_1 = x, \tilde{Y}_k = y\right]. \tag{6}$$

Besides (5), there are two other necessary conditions that $\hat{b}$ must satisfy: (i) $v^d(x,x,x) \geq \hat{b}(x)$, $\forall x \in [\underline{x},\overline{x}]$; and (ii) $\hat{b}(\underline{x}) = v^d(\underline{x},\underline{x},\underline{x})$. Condition (i) follows since expected profit for bidder 1 has to be positive in equilibrium. Condition (ii) follows from (i) and the fact that if $\hat{b}(\underline{x}) < v^d(\underline{x},\underline{x},\underline{x})$, then by slightly increasing the bid to $b(\underline{x}) + \epsilon$ when $\tilde{X}_1 = \underline{x}$, expected profit can be raised from zero to some strictly positive amount.

The solution to (5) with the boundary condition $\hat{b}(\underline{x}) = v^d(\underline{x},\underline{x},\underline{x})$ is

$$\hat{b}(x) = v^d(x,x,x) - \int_{\underline{x}}^{x} L(u|x)dt(u) + \int_{\underline{x}}^{x}\frac{h(u)}{f_k(u|u)}dL(u|x), \tag{7}$$

where

$$\begin{aligned} L(u|x) &= \exp\left\{-\int_u^x \frac{f_k(s|s)}{F_k(s|s)}ds\right\}, \\ t(u) &= v^d(u,u,u), \\ h(u) &= \int_{\underline{x}}^{u} v_1^d(u,u,y)f_k(y|u)dy. \end{aligned} \tag{8}$$

Note that $L(u|x)$ and $t(u)$ are increasing functions of u and thus are measures on $[\underline{x},\overline{x}]$. We will show in what follows that $\hat{b}(x)$ of (7) also satisfies condition (i), maximizes expected profit under the hypothesis that $(\tilde{X}_1,\ldots,\tilde{X}_n,\tilde{P})$ are *information complements* with respect to $\tilde{V}$, and is strictly increasing.

Definition 1 *Random variables, $(\tilde{Z}_1,\ldots,\tilde{Z}_m)$, are said to be information complements with respect to another random variable $\tilde{T}$ if*

$$\frac{\partial^2\phi(z_1,\ldots,z_m)}{\partial z_i\partial z_j} \geq 0, \quad \forall i \neq j,\ \forall z_1,\ldots,z_m,$$

where

$$\phi(z_1,\ldots,z_m) \equiv \mathbf{E}\left[\tilde{T}\middle|\tilde{Z}_1 = z_1,\ldots,\tilde{Z}_m = z_m\right].$$

Thus, the random variables $(\tilde{X}_1, \dots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$ if the marginal contribution to the conditional expectation of $\tilde{V}$ of a higher realization of $\tilde{X}_i$ is larger the higher the realization of any other $\tilde{X}_j$ or $\tilde{P}$. This information complementarity condition is satisfied by a large class of distributions. For example, if $\phi(z_1, \dots, z_m)$ is linear in the z_i's, then $(\tilde{Z}_1, \dots, \tilde{Z}_m)$ are information complements. Thus if $(\tilde{T}, \tilde{Z}_1, \dots, \tilde{Z}_m)$ are multivariate normally distributed, then $(\tilde{Z}_1, \dots, \tilde{Z}_m)$ are information complements with respect to $\tilde{T}$. We give three examples of strictly affiliated random variables that also satisfy the information complementarity condition.

Example 1 *Let $(\tilde{T}, \tilde{Z}_1, \dots, \tilde{Z}_m)$ be multivariate normally distributed with density function $g(t, z_1, \dots, z_m)$. Let Σ be the variance-covariance matrix of these random variables and assume that Σ^{-1} exists and has strictly negative off-diagonal elements. It is easily verified that $\partial^2 \ln g/\partial t \partial z_i > 0$ and $\partial^2 \ln g/\partial z_i \partial z_j > 0$ for $i \neq j$. Theorem 1 of Milgrom and Weber (1982a) then implies that $(\tilde{T}, \tilde{Z}_1, \dots, \tilde{Z}_m)$ are strictly affiliated. Since $\mathbf{E}[\tilde{T}|\tilde{Z}_1, \dots, \tilde{Z}_m]$ is linear in the $\tilde{Z}_i$'s, $(\tilde{Z}_1, \dots, \tilde{Z}_m)$ are information complements with respect to $\tilde{T}$.*

Besides the multivariate normally distributed random variables, there is a large class of distributions with linear conditional expectations. The following is an example.

Example 2 *Let $\tilde{Z}_i$, $i = 1, \dots, m$ be independent conditional on $\tilde{T}$ and distributed according gamma distribution given $\tilde{T} = t$:*

$$g_i(z_i|t) = \begin{cases} \frac{\left(\frac{1}{t}\right)^\alpha}{\Gamma(\alpha)} z_i^{\alpha-1} e^{-z_i/t} & \text{if } z_i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$, $t > 0$, and Γ is the gamma function. Let $1/\tilde{T}$ also be distributed according to gamma distribution with a density

$$h(1/t) = \begin{cases} \frac{\sigma^\gamma}{\Gamma(\gamma)} \left(\frac{1}{t}\right)^{\gamma-1} e^{-\sigma/t} & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma > 0$ and $\sigma > 0$. Using Theorem 1 of Milgrom and Weber (1982), one verifies that $(\tilde{T}, \tilde{Z}_1, \dots, \tilde{Z}_m)$ are strictly affiliated. Direct computation yields

$$\mathbf{E}[\tilde{T}|\tilde{Z}_1, \dots, \tilde{Z}_m] = \frac{\sum_{i=1}^m \tilde{Z}_i + \sigma}{m\alpha + \gamma - 1}.$$

Thus $(\tilde{Z}_1, \dots, \tilde{Z}_m)$ are information complements with respect to $\tilde{T}$.

Note that the prior distribution of $\tilde{T}$ in Example 2 is an element of the family of "conjugate distributions" of gamma distribution; see DeGroot (1970, Chapter 9). Other distributions with linear conditional expectations can be constructed similarly. Interested readers should consult Ericson (1969) and DeGroot (1970).

The following example gives random variables that are strict information complements.

Example 3 *Let $\tilde{Z}_i$, $i = 1, 2, \ldots, m$, be independent conditional on $\tilde{T}$ with density*

$$g_i(z_i|t) = \begin{cases} \frac{z_i^{at}}{at+1} & \text{if } z_i, t \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

The density of $\tilde{T}$ is

$$h(t) = \begin{cases} \frac{(n+1)a}{(a+1)^{n+1}-1}(at+1)^n & \text{if } t \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $\partial^2 \ln g_i(z_i|t)/\partial z_i \partial t > 0 \ \forall z, t \in (0,1)$. Theorem 1 of Milgrom and Weber (1982) implies that g_i satisfies the strict affiliation inequality. The same theorem also shows that

$$g(t, z_1, z_2, \ldots, z_m) = \begin{cases} h(t)\prod_{i=1}^m g_i(z_i|t) & \text{if } z_1, z_2, \ldots, z_m, t \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

is strictly affiliated. Direct computation yields

$$\phi(z_1, z_2 \ldots, z_m) \equiv \mathrm{E}\left[\tilde{T}\middle|\tilde{Z}_1 = z_1, \tilde{Z}_2 = z_2, \ldots, \tilde{Z}_m = z_m\right] = \frac{(\prod_{i=1}^m z_i)^a}{(\prod_{i=1}^m z_i)^a - 1} - \frac{1}{a\ln(\prod_{i=1}^m z_i)},$$

for $z_1, z_2, \ldots, z_m \in (0,1)$. Finally, one can also verify that $\partial^2\phi(z_1, z_2, \ldots, z_m)/\partial z_i \partial z_j > 0$ for all $i \neq j$ if $a \in (0,1)$.

The following lemma is a direct consequence of the definition of information complementarity. All proofs are in an appendix.

Lemma 1 *Suppose that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. Then $v_1^d(x, x', y)$ is strictly increasing in x' and y, where v_1^d denotes the partial derivative of v^d with respect to its first argument.*

The following proposition shows that if $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$, then $\hat{b}$ of (7) satisfies condition (i), that is, $v^d(x, x, x) \geq \hat{b}(x) \ \forall x \in [\underline{x}, \overline{x}]$.

Proposition 1 *Suppose that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. Then $v^d(x,x,x) \geq \hat{b}(x)$, $\forall x \in [\underline{x}, \overline{x}]$ and the inequality is strict for $x > \underline{x}$, where $\hat{b}$ is defined in (7).*

A corollary of Proposition 1 is that $\hat{b}$ is strictly increasing. Thus our assumption that resale market buyers can invert the primary bids to obtain the bidders' signal realizations is justified.

Corollary 1 *The strategy $\hat{b}$ defined in (7) is strictly increasing.*

Before proceeding, we first record two lemmas that are direct consequences of the definition of affiliation.

Lemma 2 (Milgrom and Weber (1982a)) *$F_k(y|x)/f_k(y|x)$ is decreasing in x.*

Lemma 3 *Let $x' \geq x \geq y$. Then $F_k(y|x')/F_k(x|x') \leq F_k(y|x)/F_k(x|x)$. That is, the distribution function $F_k(\cdot|x')/F_k(x'|x')$ dominates the distribution function $F_k(\cdot|x)/F_k(x|x)$ in the sense of first order stochastic dominance.*

The main result of this section is

Theorem 1 *The n-tuple $(\hat{b}, \ldots, \hat{b})$, with $\hat{b}$ as defined in (7), is a Nash equilibrium of the discriminatory auction provided that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$.*

When bidders in the discriminatory auction participate only for the purpose of consumption, Milgrom and Weber (1982a) have identified a symmetric Nash equilibrium with a bidding strategy

$$b^d(x) = v^d(x,x,x) - \int_{\underline{x}}^{x} L(u|x)dt(u), \tag{9}$$

where $L(u|x)$ and $t(u)$ are as defined in (8). The bidding strategy for the purpose of resale identified in Theorem 1 is strictly higher than b^d for every $x \in (\underline{x}, \overline{x}]$ by an amount equal to

$$\int_{\underline{x}}^{x} \frac{h(u)}{f_k(u|u)} dL(u|x),$$

the magnitude of which depends on v_1^d, the responsiveness of the resale value to the submitted bid.[10] It is this informational link between the resale value and the bids submitted by bidders in the discriminatory auction that gives the bidders an incentive to signal. Of course, since the $\hat{b}$ is strictly increasing, the resale buyers can invert the bids announced by the auctioneer to obtain the private information of the bidders and, as in Ortega-Reichart (1968) and Milgrom and Roberts (1982), in equilibrium no one gets deceived.

It is worth emphasisng that the primary bidders benefit if the resale market buyers are better informed about the true value. If, for instance, $\tilde{P} \equiv \tilde{V}$ or if $\tilde{P}$ "contains" all the relevant information in $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$,, then there would be no signalling incentive for the bidders and the expected price(s) in the auction would be lower. One might expect the "informativeness" of $\tilde{P}$ to increase with the length of the time period between the end of the primary auction and the start of the resale market.

Next we establish that the symmetric equilibrium identified above is the unique symmetric equilibrium if secondary market buyers beliefs satisfy a monotonicity condition. The beliefs of the secondary market buyers about the bidders' private information are said to be *monotone* if they are nondecreasing in the bids submitted. For instance, if the secondary market buyers believe that bidder $1's$ private information lies in the interval $[x_1^l(b), x_1^u(b)]$ when bidder 1 bids b, then the monotonicity condition on beliefs would imply that $x_1^l(\cdot)$ and $x_1^u(\cdot)$ are nondecreasing functions.[11] Since bidders with higher realizations of their private information would expect higher resale prices, it is natural to expect them to bid more. Therefore monotonicity of beliefs seems to be a natural restriction to impose. Of course, when bids are in the range of the equilibrium bidding strategies, beliefs are obtained by inverting the bidding strategies.

Since a symmetric equilibrium is a natural focal point in a game with symmetric players, and as shown below $(\hat{b}, ..., \hat{b})$ is the only symmetric equilibrium in nondecreasing strategies when the resale buyers' beliefs are monotone, we will use this equilibrium when comparing expected revenues between a discriminatory auction and a uniform-price auction.

Theorem 2 *If the secondary market buyers have monotone beliefs then $(\hat{b}, \hat{b}, ..., \hat{b})$, where $\hat{b}$ is as defined in (7), is the unique symmetric equilibrium in nondecreasing strategies.*

[10] If $\tilde{P} \equiv \tilde{V}$ there is no signalling motive and the equilibrium strategy is as specified in (9), since $r^d(\cdot) \equiv \tilde{V}$ and thus $v_1^d(\cdot) \equiv 0$ and $h(\cdot) \equiv 0$. Also, when $\tilde{P}$ is "very informative" about $\tilde{V}$ the signalling motive is weak. For example if we let $\tilde{P}_n \equiv \tilde{V} + \epsilon_n$ where ϵ_n is, say, uniformly distributed on $[-\frac{1}{n}, \frac{1}{n}]$ then as n increases the resale price becomes less responsive to the players' bids and in the limit the incentive to signal disappears.

[11] In the symmetric equilibrium of Theorem 1, the resale buyers beliefs are monotone (since $\hat{b}$ is increasing) with $x_1^l(b) = x_1^u(b) = \hat{b}^{-1}(b)$. For $b > \hat{b}(\overline{x})$, $x_1^l(b) = x_1^u(b) = \overline{x}$, and for $b < \hat{b}(\underline{x})$, $x_1^l(b) = x_1^u(b) = \underline{x}$.

3.2 Public announcement of the auctioneer's information

Suppose that the auctioneer has private information about $\tilde{V}$, represented by a random variable $\tilde{X}_0$. We will consider the impact of announcing $\tilde{X}_0$ before the auction on the expected selling price. Let $\bar{b}(\cdot; x_0)$ be a symmetric equilibrium bidding strategy conditional on $\tilde{X}_0 = x_0$. It is assumed that $\bar{b}(\cdot; x_0)$ is increasing and differentiable in x. If bidder 1 bids b and wins then the resale price in this case will be

$$p^d(\bar{b}^{-1}(b; x_0), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}; x_0) \equiv \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 = \bar{b}^{-1}(b; x_0), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}, \tilde{X}_0 = x_0\right],$$

where $\bar{b}(\bar{b}^{-1}(b; x_0); x_0) = b$. Putting

$$w^d(x', x, y; x_0) \equiv \mathbf{E}\left[p^d(x', \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}; x_0) \middle| \tilde{X}_1 = x, \tilde{Y}_k = y, \tilde{X}_0 = x_0\right],$$

it is straightforward to show that $\bar{b}(x; x_0)$ must satisfy

$$\bar{b}'(x; x_0) = (w^d(x, x, x; x_0) - \bar{b}(x; x_0))\frac{f_k(x|x; x_0)}{F_k(x|x; x_0)} + \int_{\underline{x}}^{x} w_1^d(x, x, y; x_0) f_k(y|x; x_0) dy. \quad (10)$$

where $f_k(y|x; x_0)$ denotes the conditional density of $\tilde{Y}_k$ given $\tilde{X}_1 = x$ and $\tilde{X}_0 = x_0$. In addition, the boundary condition $\bar{b}(\underline{x}; x_0) = w^d(\underline{x}, \underline{x}, \underline{x}; x_0)$ must be satisfied. The solution to (10) with this boundary condition is

$$\bar{b}(x; x_0) = w^d(x, x, x; x_0) - \int_{\underline{x}}^{x} L(u|x; x_0) dt(u; x_0) + \int_{\underline{x}}^{x} \frac{h(u; x_0)}{f_k(u|u; x_0)} dL(u|x; x_0), \quad (11)$$

where

$$\begin{aligned}
L(u|x; x_0) &= \exp\left\{-\int_u^x \frac{f_k(s|s; x_0)}{F_k(s|s; x_0)} ds\right\}, \\
t(u; x_0) &= w^d(u, u, u; x_0), \\
h(u; x_0) &= \int_{\underline{x}}^{u} w_1^d(u, u, y; x_0) f_k(y|u; x_0) dy.
\end{aligned}$$

Next, we define conditional information complements.

Definition 2 *Random variables, $(\tilde{Z}_1, \ldots, \tilde{Z}_m)$, are said to be information complements conditional on random variable $\tilde{Y}$ with respect to random variable $\tilde{T}$ if*

$$\frac{\partial^2 \phi(z_1, \ldots, z_m, y)}{\partial z_i \partial z_j} \geq 0, \quad \forall i \neq j, \ \forall z_1, \ldots, z_m, \forall y,$$

where

$$\phi(z_1, \ldots, z_m, y) \equiv \mathbf{E}\left[\tilde{T} \middle| \tilde{Z}_1 = z_1, \ldots, \tilde{Z}_m = z_m, \tilde{Y} = y\right].$$

If $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ are information complements conditional on $\tilde{X}_0$ with respect to $\tilde{V}$, then a proof identical to that of Theorem 1 shows that $\bar{b}$ defined in (11) is a symmetric equilibrium strategy. This is stated without proof in the following proposition.

Proposition 2 *The n-tuple $(\bar{b}(\cdot; x_0), \ldots, \bar{b}(\cdot; x_0))$, with $\bar{b}(\cdot; x_0)$ as defined in (11), is a Nash equilibrium of the discriminatory auction when the auctioneer announces $\tilde{X}_0 = x_0$, provided that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements conditional on $\tilde{X}_0$ with respect to $\tilde{V}$ and the resale market participants believe that all the bidders follow strategy $\bar{b}(\cdot; x_0)$.*

Note that the existence of an equilibrium does not depend on whether $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. There exists a Nash equilibrium as long as $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ are information complements conditional on $\tilde{X}_0$ with respect to $\tilde{V}$. Our main result in this subsection will be that the expected selling price under the policy of always reporting $\tilde{X}_0$ cannot be lower than that under any other reporting policy provided that $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$.[12] We first show, in the next proposition, that $\bar{b}(x; x_0)$ is a increasing function of x_0.

Proposition 3 *Suppose that $(\tilde{V}, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are affiliated and that $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. Then $\bar{b}(x; x_0)$ is an increasing function of x_0.*

The main result of this section is

Theorem 3 *Suppose that $(\tilde{V}, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are affiliated and that $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$. A policy of publicly revealing the seller's information cannot lower, and may raise, the expected revenue for the seller in a discriminatory auction.*

Given Proposition 3, the proof of theorem 3 mimics that of Milgrom and Weber (1982a, Theorem 16), and is omitted. The interested reader is referred to Bikhchandani and Huang (1988) for details.

Theorem 3 depends critically on the fact that under its hypothesis $\bar{b}(x; x_0)$ is increasing in x_0. When $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are not information complements, $\bar{b}(x; x_0)$ may not be

[12]Note that this assumption is stronger than the conditional information complementarity required for existence of Nash equilibrium.

an increasing function of x_0, and revealing $\tilde{X}_0$ may reduce the bidders' incentive to signal. This in turn may lower the expected revenue for the seller even though $(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are affiliated.

4 Uniform-price auction

In a uniform-price auction the bidders submit sealed bids and the k highest bidders win the auction. The price they pay is equal to the $(k+1)$st highest bid. Initially we obtain a candidate for a symmetric Nash equilibrium under the assumption that after the auction the auctioneer reveals the winning bids and the highest losing bid, that is, the price paid by the winning bidders. If any of the lower bids are also revealed, our results remain unchanged.

We first obtain strategies that satisfy first-order necessary conditions for a symmetric Nash equilibrium. Next we show that when there exists a symmetric equilibrium in the uniform-price auction, the auctioneer's expected revenues at this equilibrium are greater than at the symmetric equilibrium of the discriminatory auction. The intuition behind this result is as follows. From the theory of auctions without resale markets we know that compared with a discriminatory auction, a greater amount of information is revealed during a uniform-price auction. This weakens the winners' curse in the uniform-price auction and results in greater revenues for the auctioneer. In addition when the primary auction and the resale markets are informationally linked, as in our model, it is cheaper to submit higher bids in the uniform-price auction in order to signal to the resale market buyers. This in turn further increases the expected revenues from uniform-price auctions.

However, the second of these two factors — the fact that in a uniform-price auction the price paid by a bidder conditional upon winning does not increase as his bid is increased — can result in nonexistence of equilibrium in a uniform-price auction. If the resale price is very responsive to the bids submitted there may exist an incentive for the bidders to submit arbitrarily large bids and upset any purported equilibrium. In the last part of this section we show by example that this is indeed possible, and more generally show that when $k = 1$, and $\tilde{P}$ is a constant there does not exist any Nash equilibrium in strictly increasing pure strategies.

4.1 Necessary conditions for a symmetric equilibrium

As in the discriminatory auction, each primary bidder's strategy is a function from $[\underline{x}, \overline{x}]$ to the real line. Suppose that $(b_0, b_0, \ldots, b_0)$ is a Nash equilibrium in strictly increasing and differentiable strategies, when buyers in the secondary market believe that all bidders use b_0. If all other bidders use strategy b_0, bidder 1 receives information $\tilde{X}_1 = x$ and submits a bid equal to b, then if bidder 1 wins the resale price is

$$\begin{aligned} r^u(b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) &\equiv \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 = b_0^{-1}(b), b_0^{-1}(b_0(\tilde{Y}_1)), \ldots, b_0^{-1}(b_0(\tilde{Y}_k), \tilde{P}\right] \\ &= \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 = b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}\right]. \end{aligned} \tag{12}$$

r^u is strictly increasing in all its arguments. Note that $r^u(\cdot) \equiv r^d(\cdot)$. If bidder 1 wins the auction, the expected resale price conditional on $\tilde{X}_1$ and $\tilde{Y}_k$ is

$$v^u(b_0^{-1}(b), x, y) \equiv \mathbf{E}\left[r^u(b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) \middle| \tilde{X}_1 = x, \tilde{Y}_k = y\right]. \tag{13}$$

By strict affiliation, v^u is strictly increasing in its arguments. From the definition of v^d it follows that

$$v^u(x', x, y) = v^d(x', x, y) \quad \forall x', x, y. \tag{14}$$

We show below that b_0 must be given by the following equation:

$$b_0(x) = v^u(x, x, x) + \frac{h(x)}{f_k(x|x)}, \tag{15}$$

where $h(x)$ is as defined in (8).

If $X_1 = x$ and bidder 1 bids b, his expected profit is

$$\begin{aligned} \Pi^u(b|x) &\equiv \mathbf{E}\left[\left(r^u(b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b_0(\tilde{Y}_k)\right) 1_{\{b \geq b_0(\tilde{Y}_k)\}} \middle| \tilde{X}_1 = x\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\left(r^u(b_0^{-1}(b), \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b_0(\tilde{Y}_k)\right) 1_{\{b \geq b_0(\tilde{Y}_k)\}} \middle| \tilde{X}_1, \tilde{Y}_k\right] \middle| \tilde{X}_1 = x\right] \\ &= \mathbf{E}\left[\left(v^u(b_0^{-1}(b), \tilde{X}_1, \tilde{Y}_k) - b_0(\tilde{Y}_k)\right) 1_{\{b \geq b_0(\tilde{Y}_k)\}} \middle| \tilde{X}_1 = x\right] \\ &= \int_{\underline{x}}^{b_0^{-1}(b)} (v^u(b_0^{-1}(b), x, y) - b_0(y)) f_k(y|x) dy. \end{aligned} \tag{16}$$

Taking the first derivative of $\Pi^u(b|x)$ with respect to b gives

$$\begin{aligned} \tfrac{\partial \Pi^u(b|x)}{\partial b} &= \left(v^u(b_0^{-1}(b), x, b_0^{-1}(b)) - b\right) f_k(b_0^{-1}(b)|x)(b_0'(b_0^{-1}(b)))^{-1} \\ &\quad + (b_0'(b_0^{-1}(b)))^{-1} \textstyle\int_{\underline{x}}^{b_0^{-1}(b)} v_1^u(b_0^{-1}(b), x, y) f_k(y|x) dy, \end{aligned} \tag{17}$$

where $b_0'(x)$ is the derivative of $b_0(x)$ and v_1^u is the partial derivative of v^u with respect to its first argument. For $(b_0, \ldots, b_0)$ to be a Nash equilibrium, it is necessary that relation (17) be zero when $b = b_0(x)$. That is,

$$0 = b_0'(x) \frac{\partial \Pi^u(b|x)}{\partial b}\Big|_{b=b_0(x)} = (v^u(x,x,x) - b_0(x)) f_k(x|x) + \int_{\underline{x}}^{x} v_1^u(x,x,y) f_k(y|x) dy. \quad (18)$$

where we use the assumption that b_0 is strictly increasing. Rearranging terms implies that $b_0(x)$ is as defined in (15).

In a uniform-price auction without resale markets Milgrom and Weber (1982a) show that the symmetric Nash equilibrium bidding strategy is $b^u(x) = v^u(x,x,x)$. As in discriminatory auctions, the bidding strategy b_0 is strictly higher than b^u, for every $x \in (\underline{x}, \overline{x}]$, by an amount which depends on v_1^u, the responsiveness of the resale value to the submitted bid.

4.2 Revenue comparison with the discriminatory auction

Next we show that when there exists a symmetric equilibrium in the uniform-price auction, it generates strictly greater expected revenues than the symmetric equilibrium of the discriminatory auction.[13]

Theorem 4 *When $(b_0, ..., b_0)$ is a symmetric Nash equilibrium in the uniform-price auction, the expected revenues generated at this equilibrium are strictly greater than the expected revenue at the symmetric equilibrium of the discriminatory auction.*

4.3 Possibility of nonexistence of equilibrium

In this subsection we illustrate the possibility that strong signalling incentives on the part of the bidders may lead to nonexistence of a pure strategy Nash equilibrium when winning bids are announced in a uniform-price auction. First we present an example in which $\max\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P}\}$ is a sufficient statistic of $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ for the posterior density of $\tilde{V}$. Although in this example the random variables are only weakly affiliated, it illustrates the difficulties that arise when the resale price is very responsive to the winning bids.

[13] An argument similar to that in Theorem 2 establishes that b_0 is the only candidate for a symmetric equilibrium when the resale market buyers have monotone beliefs. This remark also applies to the symmetric equilibrium we will obtain in Section 5.

Example 4 *Suppose that all the bids are announced after a uniform-price auction. The prior marginal density of* $\tilde{V}$ *is uniform with support* $[0,1]$. *The random variables* $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P})$ *are identically distributed, are independent conditional on* $\tilde{V}$, *and their conditional density is uniform on* $[0, \tilde{V}]$. *Let* $\tilde{Z} = \max\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P}\}$. *It is readily confirmed that* $\mathbf{E}[\tilde{V}|\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \tilde{P}] = \mathbf{E}[\tilde{V}|\tilde{Z}]$. *Clearly, the resale price will be equal to* $\mathbf{E}[\tilde{V}|\tilde{Z}]$, *and*

$$\begin{aligned} \mathbf{E}[\tilde{V}|\tilde{Z}] &\geq \tilde{Z} \\ \mathbf{E}[\tilde{V}|\tilde{Z}=1] &= 1. \end{aligned} \tag{19}$$

Let $(b_1, b_2, \ldots, b_n)$ *be a candidate Nash equilibrium, with* b_i *strictly increasing. We limit our attention to weakly undominated strategies and thus* $0 \leq b_i(\tilde{X}_i) \leq 1$. *Let* $\Pi^u(b|x)$ *be bidder* 1*'s expected payoff when he bids* b *and* $\tilde{X}_1 = x$. *Then it is easily verified that if* $x < \overline{x}$ *then*

$$\Pi^u(b_1(x)|x) \leq \Pi^u(b_1(\overline{x})|x), \tag{20}$$

since if bidder 1 bids $b_1(\overline{x})$, *he will always win whenever he would have with a bid of* $b_1(x)$ *and pay the same price. In addition he will also win whenever the k-th highest bid of the others' bid is in* $(b_1(x), b_1(\overline{x}))$. *Moreover, since* b_1 *is strictly increasing, (19) implies that the resale price if he bids* $b_1(\overline{x})$ *is equal to one which is at least as large as the resale price if he wins with a bid of* $b_1(x)$. *The inequality in (20) is strict as long as there is a nonzero probability that the k-th highest bid is in the interval* $(b_1(x), b_1(\overline{x}))$.

Thus the only candidate Nash equilibrium appears to be a somewhat degenerate one in which at least one of the bidders, say bidder 1, always bids one, and at least $n-k$ *bidders bid sufficiently low so that they never win, and the* $n-k$ *lowest bid is low enough so that bidder 1 always maximizes his profits by bidding one. But if there is any strictly positive cost of participating in the auction (such as an entry fee or a bid preparation cost) then the* $n-k$ *bidders who never win at this equilibrium will not participate in the auction.*

Next we show that if $k = 1$ and if $\tilde{P}$ is totally uninformative about $\tilde{V}$, then there does not exist a symmetric Nash equilibrium in strictly increasing strategies. Essentially when there is only one object, and no other information becomes public after the auction, there exists a large incentive for the bidders to submit high bids, since the resale price is very responsive to the winning bid.

Proposition 4 *Suppose that $k = 1$ and that $\tilde{P}$ is independent of $\tilde{V}$. Then if the winning bid and the highest losing bid are announced there does not exist a Nash equilibrium in strictly increasing strategies.*

However, there exist other examples for which an equilibrium exists. For instance if $\tilde{V}$ is uniform on $[0,1]$ and $\tilde{X}_1$ is uniform on $[\tilde{V},1]$, then $v_1^u(\cdot,\cdot,\cdot) \equiv 0$, and the equilibrium strategy is to bid $b_0(x) = v^u(x,x,x)$. Whether there exist intuitive sufficient conditions under which an equilibrium exists remains an open question.

5 Uniform-price auction without signalling

Since there exists a possibility of nonexistence of equilibrium in a uniform-price auction with signalling, in this section we analyze uniform-price auctions when the winning bids are not announced. Thus the bidders do not have an incentive to signal their private information, since if they win their bids are not revealed. Even without the signalling incentive, we are able to show that in at least two scenarios the expected revenue generated by this uniform-price auction is higher than that generated by the discriminatory auction discussed in Section 3. We believe that the first scenario is a plausible one for the case of the Treasury bill market.

5.1 Existence of a symmetric Nash equilibrium when the winning bids are not announced

We show below that there exists a symmetric Nash equilibrium in strictly increasing and differentiable strategies, $(b^*, b^*, \ldots, b^*)$, when buyers in the secondary market believe that all bidders use b^*. Suppose that bidders $i = 2, \ldots, n$ adopt the strategy b^*, bidder 1 receives information $\tilde{X}_1 = x$, and submits a bid equal to b. If bidder 1 wins the resale price is

$$\begin{aligned} \hat{r}^u(\tilde{Y}_k, \tilde{P}) &\equiv \mathbf{E}\left[\tilde{V} \middle| \tilde{Z}_{k+1} = \tilde{Y}_k, \tilde{P}\right] \\ &= \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k, \tilde{P}\right], \end{aligned}$$

where $\tilde{Z}_j$ is the j-th order statistic of $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$. The equality follows from the fact that the signals are identically distributed. $\hat{r}^u$ is strictly increasing in both its arguments. If bidder 1 wins the auction, the expected resale price conditional on $\tilde{Y}_k$ and $\tilde{X}_1$ is

$$\hat{v}^u(x,y) \equiv \mathbf{E}\left[\hat{r}^u(\tilde{Y}_k, \tilde{P}) \middle| \tilde{X}_1 = x, \tilde{Y}_k = y\right].$$

By strict affiliation, $\hat{v}^u$ is strictly increasing in its arguments. Thus, if $X_1 = x$ and bidder 1 bids b, his expected profit is

$$
\begin{aligned}
\hat{\Pi}^u(b|x) &\equiv \mathbf{E}\left[\left(\hat{r}^u(\tilde{Y}_k, \tilde{P}) - b^*(\tilde{Y}_k)\right) 1_{\{b \geq b^*(\tilde{Y}_k)\}} \middle| X_1 = x\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\left(\hat{r}^u(\tilde{Y}_k, \tilde{P}) - b^*(\tilde{Y}_k)\right) 1_{\{b \geq b^*(\tilde{Y}_k)\}} \middle| \tilde{X}_1, \tilde{Y}_k\right] \middle| \tilde{X}_1 = x\right] \\
&= \mathbf{E}\left[\left(\hat{v}^u(\tilde{X}_1, \tilde{Y}_k) - b^*(\tilde{Y}_k)\right) 1_{\{b \geq b^*(\tilde{Y}_k)\}} \middle| \tilde{X}_1 = x\right]. \qquad (21)
\end{aligned}
$$

Define

$$b^*(x) \equiv \hat{v}^u(x, x). \qquad (22)$$

Note that b^* is strictly increasing. We show that $(b^*, b^*, \ldots, b^*)$ is an equilibrium.

Theorem 5 *The n-tuple $(b^*, b^* \ldots, b^*)$ is a Nash equilibrium in the uniform-price auction provided that resale market buyers believe that all the bidders follow the strategy b^*.*

Milgrom and Weber (1982a) have shown that the price paid by winning bidders in a uniform-price auction when bidders participate for the purposes of consumption is $\mathbf{E}[\tilde{V}|\tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k]$. We show in the next lemma that the price paid by winning bidders in the uniform-price auction in our model is greater than this. This is true even though the primary bidders do not have a signalling motive.

Lemma 4 *With probability one, the price paid by winning bidders in a uniform-price auction with a resale market is greater than that in a uniform-price auction without resale markets (in which the bidders participate for consumption). That is*

$$b^*(\tilde{Y}_k) > \mathbf{E}[\tilde{V} \big| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k]$$

with probability one.

The "true value" of the object for the primary bidders is the resale price. Thus, if no additional information becomes available after the auction, that is if $\tilde{P}$ is constant, the winners' curse on the primary bidders is weakened. Since there is no signalling motive, one would expect the bids in the primary auction to increase when $\tilde{P}$ is constant (or when $\tilde{P}$ is independent of $(\tilde{V}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$). This is proved in the next lemma.

Lemma 5 *With probability one, the bids in the uniform-price auction strictly increase when $\tilde{P}$ is constant, that is when no additional information (other than about the bids submitted in the auction) becomes public after the auction.*

5.2 Revenue comparison with the discriminatory auction

We obtain two sets of sufficient conditions under which the expected revenues generated at the symmetric equilibrium of the uniform-price auction obtained in the previous subsection, are greater than the expected revenues at the symmetric equilibrium of the discriminatory auction of Section 3. The first set seems plausible for the case of Treasury bill auctions.

The following theorem states that if the public information, $\tilde{P}$, is not very informative about the true value of the objects, the uniform-price auction generates higher expected revenue than the discriminatory auction.

Theorem 6 *There exists a scalar $M > 0$ such that if $\partial \hat{r}^u(y,p)/\partial p \leq M$ for all $y, p \in [\underline{x}, \overline{x}] \times [\underline{p}, \overline{p}]$, then the uniform-price auction without signalling generates strictly higher expected revenue than the discriminatory auction with signalling, at their respective symmetric equilibria.*

In words, Theorem 6 says that if the *ex post* public information $\tilde{P}$ has little impact on the resale price conditional on the information released from the uniform-price auction, then the auctioneer's expected revenue is higher in the uniform-price auction even when winning bids are not announced. This is true even though there is no signalling aspect in the uniform-price auction. In the case of the Treasury bill auction, bids are submitted before 1:00pm every Monday. The results of the auction are announced around 4:30pm and the resale market comes into play. One would expect that any public information that normally arrives between 1:00pm and 4:30pm would not be very informative about $\tilde{V}$ conditional on the results of the earlier auction.

The following theorem gives an alternative scenario under which once again the uniform-price auction generates higher revenues. Essentially it says that if the signalling motive of the bidders is not strong, then the uniform-price auction generates higher expected revenue.

Theorem 7 *Suppose that $(\tilde{V}, \tilde{X}_1, \ldots, \tilde{X}_n)$ are strictly affiliated. There exists a scalar $M > 0$ such that if $\partial r^d(x, y_1, \ldots, y_k, p)/\partial x \leq M$ for all $x, y_1, \ldots, y_k, p \in [\underline{x}, \overline{x}]^{k+1} \times [\underline{p}, \overline{p}]$, then the uniform-price auction without signalling generates strictly higher expected revenue than the discriminatory auction with signalling, at their respective symmetric equilibria.*

A scenario where Theorem 7 is applicable is when the public information $\tilde{P}$ is very informative about the true value $\tilde{V}$. Then the impact of $\tilde{X}_1$ on the resale price will be small

when bidder 1 wins. This scenario, however, does not seem to be plausible in the case of Treasury bill auctions.

6 Concluding remarks

This paper is an exploratory study of competitive bidding when there exists a resale market which is informationally linked to the bidding. We have shown that there exists a symmetric Nash equilibrium in discriminatory auctions when winning bids and the highest losing bid are announced provided that the relevant variables are affiliated and are information complements. This is the only symmetric equilibrium when the resale market buyers have monotone beliefs. If his information is complementary to that of the bidders, the auctioneer will increase his expected revenue by precommitting to announce his private information before the auction. We know little about the general impact of the *ex post* information $\tilde{P}$ on the signalling motive of bidders. For the case of Treasury bill markets this is not important since we believe that very little additional information becomes publicly available in the short time period between the closing of the Treasury bill auction and the opening of the resale market. A related question which is of greater importance for Treasury bill auctions is whether there exist plausible scenarios in which the auctioneer can increase expected revenue by announcing his private information after the auction (and before the secondary markets convene) rather than before the auction. These warrant further investigation.

We established the possibility of nonexistence of an equilibrium in a uniform-price auction when winning bids are announced. However, when there exists a symmetric equilibrium, it generates greater expected revenues than the symmetric equilibrium in a discriminatory auction. We also showed that there exists a symmetric Nash equilibrium in a uniform-price auction when only the highest losing bid is announced, so that there is no signalling incentive for the bidders. Two scenarios are provided where the uniform-price auction without signalling generates strictly higher expected revenue for the auctioneer than the discriminatory auction.

Some implications of our model deserve attention. First, it is easy to incorporate noncompetitive bids in our model, provided the total amount of noncompetitive bids, j, is common knowledge before the auction. We can model the primary auction as one with n bidders and $k+j$ objects, j of which are awarded to noncompetitive bidders at the average price. The preceding analysis remains unchanged. The expected profits of the noncom-

petitive bidders is positive and equal to the *ex ante* expected profits of the competitive bidders. There is a prespecified minimum and maximum quantity for each noncompetitive bid, which may be the reason that resale market buyers do not buy Treasury bills through noncompetitive bids; and it may also explain why competitive bidders do not submit only noncompetitive bids while avoiding (presumable costly) information collection.[14] Second, the fact that *ex ante* expected profits of the bidders is strictly positive (except in uniform-price auctions without signalling, when $\tilde{P}$ is uninformative about $\tilde{V}$) implies that in expectation, the average price in the auction is strictly less than the resale price. This comparision has been empirically documented by Cammack (1986). Third, as pointed out in section 3.1, the primary bidders benefit if the resale market buyers are better informed about the true value, since it decreases the bidders' signalling incentive.

There are several possible extensions of our model. First, in the Treasury bill auction, the primary bidders submit price-quantity pairs and demand more than one unit of the Treasury bill. This feature is missing in our model. Second, two weeks before the conduct of the weekly Treasury bill auction, forward contracts of the Treasury bills to be auctioned are traded among the primary bidders. The relationship among the forward prices, bids submitted, and the resale price needs to be investigated. Third, there exists a wide variety of close substitutes of Treasury bills carried as inventories by primary bidders. These close substitutes may have a significant effect on the interplay between the forward markets and the weekly auction. We hope to investigate some of these issues in our future research.

[14] By renormalizing the units, we can assume that each competitive bidder demands exactly $m > 1$ or zero units. This allows us to keep the prespecified maximum demand of noncompetitive bidders at a level less than that of competitive bidders.

7 Appendix

PROOF OF LEMMA 1: The joint density of $(\tilde{V}, \tilde{P}, \tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_{n-1})$ is

$$(n-1)!f(v,p,x,y_1,\ldots,y_{n-1})1_{\{y_1 \geq y_2 \geq \cdots \geq y_{n-1}\}}.$$

As a consequence, the conditional density of $\tilde{V}$ given $(\tilde{P}, \tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_{n-1})$ is

$$\frac{f(v,p,x,y_1,\ldots,y_{n-1})}{f(p,x,y_1,\ldots,y_{n-1})}1_{\{y_1 \geq y_2 \geq \cdots \geq y_{n-1}\}}.$$

Thus $(\tilde{P}, \tilde{X}_1, \tilde{Y}_1, \ldots., \tilde{Y}_k)$ are information complements. Let r_1^d denote the derivative of r^d, which is defined in (1), with respect to its first argument. It is then easily verified that $r_1^d(x, y_1, \ldots, y_k, p)$ is a strictly increasing function of p, y_j, $\forall j$. Next note that

$$v_1^d(x,x',y) = \mathbf{E}[r_1^d(x, \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) \big| \tilde{X}_1 = x', \tilde{Y}_k = y].$$

Theorem 5 of Milgrom and Weber (1982a) then implies, by affiliation, that $v_1^d(x,x',y)$ is a strictly increasing function of x' and y. ∎

PROOF OF PROPOSITION 1: We first write

$$\begin{aligned} v^d(x,x,x) & - \hat{b}(x) = \textstyle\int_{\underline{x}}^{x} L(u|x)dt(u) - \int_{\underline{x}}^{x} \frac{h(u)}{f_k(u|u)} dL(u|x) \\ & = \textstyle\int_{\underline{x}}^{x} L(u|x) \left(v_1^d(u,u,u) + v_2^d(u,u,u) + v_3^d(u,u,u) - \frac{h(u)}{F_k(u|u)}\right) du. \end{aligned} \tag{23}$$

By the hypothesis that $(\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{P})$ are information complements with respect to $\tilde{V}$ and Lemma 1, $v_1^d(u,u,u) \geq v_1^d(u,u,y)$ for $y \leq u$. It follows that

$$\frac{h(u)}{F_k(u|u)} = \int_{\underline{x}}^{u} v_1^d(u,u,y) \frac{f_k(y|u)}{F_k(u|u)} dy \leq v_1^d(u,u,u).$$

Substituting this relation into (23) gives

$$v^d(x,x,x) - \hat{b}(x) \geq \int_{\underline{x}}^{x} L(u|x) \left(v_2^d(u,u,u) + v_3^d(u,u,u)\right) du \geq 0.$$

Note that the above inequality is strict for $x \in (\underline{x}, \overline{x}]$ since $v_2^d > 0$ and $v_3^d > 0$ by strict affiliation. ∎

PROOF OF COROLLARY 1: We will show that $\hat{b}'(x) > 0 \; \forall x > \underline{x}$. From Proposition 1 we have $v^d(x,x,x) - \hat{b}(x) > 0 \; \forall x > \underline{x}$. The proof is completed by inserting this in (5), and noting that $v_1^d > 0$. ∎

PROOF OF LEMMA 3: By affiliation we have for $\beta \geq \alpha$, $x' \geq x$

$$f_k(\alpha|x')f_k(\beta|x) \leq f_k(\alpha|x)f_k(\beta|x').$$

Thus for $x \geq y$

$$\int_y^x \int_{\underline{x}}^y f_k(\alpha|x')f_k(\beta|x)d\alpha d\beta \leq \int_y^x \int_{\underline{x}}^y f_k(\alpha|x)f_k(\beta|x')d\alpha d\beta,$$

which is equivalent to

$$F_k(y|x')(F_k(x|x) - F_k(y|x)) \leq F_k(y|x)(F_k(x|x') - F_k(y|x')).$$

Rearranging terms gives

$$\frac{F_k(y|x')}{F_k(x|x')} \leq \frac{F_k(y|x)}{F_k x|x)}.$$

∎

PROOF OF THEOREM 1: Let $x' \leq x$. Recall from (4) that

$$\begin{aligned}
0 = \frac{\partial \Pi^d(\hat{b}(x')|x')}{\partial b} &= (\hat{b}'(x'))^{-1}F_k(x'|x')\Big((v^d(x',x',x') - \hat{b}(x'))\frac{f_k(x'|x')}{F_k(x'|x')} - \hat{b}'(x') \\
&\quad + \int_{\underline{x}}^{x'} v_1^d(x',x',y)\frac{f_k(y|x')}{F_k(x'|x')}dy\Big) \\
&\leq (\hat{b}'(x'))^{-1}F_k(x'|x')\Big((v^d(x',x,x') - \hat{b}(x'))\frac{f_k(x'|x)}{F_k(x'|x)} - \hat{b}'(x') \\
&\quad + \int_{\underline{x}}^{x'} v_1^d(x',x,y)\frac{f_k(y|x')}{F_k(x'|x')}dy\Big) \\
&\leq (\hat{b}'(x'))^{-1}F_k(x'|x')\Big((v^d(x',x,x') - \hat{b}(x'))\frac{f_k(x'|x)}{F_k(x'|x)} - \hat{b}'(x') \\
&\quad + \int_{\underline{x}}^{x'} v_1^d(x',x,y)\frac{f_k(y|x)}{F_k(x'|x)}dy\Big) \\
&= \frac{F_k(x'|x')}{F_k(x'|x)}\frac{\partial \Pi^d(\hat{b}(x')|x)}{\partial b},
\end{aligned}$$

where the first inequality follows from Proposition 1, Lemma 1, and Lemma 2, and the second inequality follows from Lemma 3. That is, when $\tilde{X}_1 = x$ and bidder 1 bids $b = \hat{b}(x') \leq \hat{b}(x)$, his expected profit can be raised by bidding higher. Similar arguments show that $\frac{\partial \Pi^d(b(x')|x)}{\partial b} \leq 0$ for $x' \geq x$. As a consequence, $\Pi^d(b|x)$ is maximized at $b = \hat{b}(x)$. Finally, since $\Pi^d(\hat{b}(\underline{x})|x) = 0$ for all x, we have $\Pi^d(\hat{b}(x)|x) > 0$ for all $x > \underline{x}$, by strict affiliation. We have thus shown that $\hat{b}(x)$ is the best strategy for bidder 1 when he observes $\tilde{X}_1 = x$, when

bidders $i = 2, 3, \ldots, n$ follow $\hat{b}$, and when the resale market participants believe that all the bidders follow $\hat{b}$. ∎

PROOF OF THEOREM 2: Suppose $(b_s, b_s, ..., b_s)$ is a symmetric equilibrium in nondecreasing strategies. Therefore b_s is differentiable almost everywhere. Thus, as in Milgrom and Weber (1982a), Theorem 14, the proof is complete once we establish that b_s must be strictly increasing and continuous. Hence b_s must satisfy the differential equation (5) and the only solution to this is $\hat{b}$ of (7).

First, suppose that b_s is not strictly increasing. That is, there exist $x_a < x_b$ such that $b_s(x) = c,\ \forall x \in [x_a, x_b]$. The resale price if bidder 1 wins with a bid b is

$$r^s(b, \tilde{B}_1, \ldots, \tilde{B}_k, \tilde{P}) = \mathbf{E}\left[\tilde{V} \middle| \tilde{b}_1 = b, \tilde{B}_1, \ldots, \tilde{B}_k, \tilde{P}\right],$$

where $\tilde{b}_1$ is the random variable which denotes bidder $1's$ bid and $\tilde{B}_l$ is the l-th order statistic of the others equilibrium bids, $(b_s(\tilde{X}_2), \ldots, b_s(\tilde{X}_n))$. Since b_s is nondecreasing, B_l is affiliated with all the other random variables in the model. Analogous to v^d we define the conditional expected resale price if bidder 1 wins

$$v^s(b, x, \beta) \equiv \mathbf{E}\left[r^s(b, \tilde{B}_1, \ldots, \tilde{B}_k, \tilde{P}) \middle| \tilde{X}_1 = x, \tilde{B}_k = \beta\right].$$

Affiliation and monotone beliefs of the resale market buyers implies that r^s and v^s are nondecreasing in all arguments. The expected profit for bidder 1 when $\tilde{X}_1 = x \in [x_a, x_b]$, and he submits an equilibrium bid $b_s(x) = c$ is

$$\begin{aligned} \Pi^s(c|x) &= \mathbf{E}\left[\left(v^s(c, \tilde{X}_1, \tilde{B}_k) - c\right) 1_{\{\tilde{B}_k < c\}} \middle| \tilde{X}_1 = x\right] \\ &+ \text{Prob}\{\text{bidder 1 wins} | \tilde{B}_k = c, \tilde{b}_1 = c\} \mathbf{E}\left[\left(v^s(c, \tilde{X}_1, \tilde{B}_k) - c\right) 1_{\{\tilde{B}_k = c\}} \middle| \tilde{X}_1 = x\right], \end{aligned} \tag{24}$$

where the probability of bidder 1 being declared a winner given that there is a tie is the expectation of the number of objects left after those who bid more than c have been assigned objects divided by the number of bidders (including bidder 1) who bid c. Since $b_s(x) = c,\ \forall x \in [x_a, x_b]$, both this probability and the probability of the event $\{\tilde{B}_k = c\}$ is strictly positive (and strictly less than one). We must have $v^s(c, x_a, c) \geq c$ else a bid $b_s(x_a) = c$ results in negative profits when $\tilde{X}_1 = x_a$. Therefore, by strict affiliation, for any $x_0 \in (x_a, x_b]$ there exists $\epsilon_1 > 0$ (which depends on x_0) such that $v^s(c, x_0, c) > c + \epsilon_1$. Thus the second expression in (24) is strictly positive for $x = x_0$. If instead of c bidder 1 bids slightly more when $\tilde{X}_1 = x_0$, there is a discontinuous jump in his probability of winning

since if $\tilde{B}_k = c$, bidder 1 will now win with probability one. Since beliefs are monotone, $v^s(c', x_0, \cdot) \geq v^s(c, x_0, \cdot)$, $\forall c' > c$. Therefore, there exists $\epsilon_2 > 0$ such that a bid of $b_s(x_0) + \epsilon_2$ leads to greater expected profits than $b_s(x_0)$. This contradicts our assumption that b_s is an equilibrium strategy. Therefore b_s must be strictly increasing.

Next suppose that b_s has a discontinuity at x_d. Consider the case where $\lim_{x \uparrow x_d} b_s(x) < b_s(x_d)$. Since b_s is strictly increasing and can be inverted we will write $v^s(x', x, y)$ instead of $v^s(b, x, \beta)$ where $b_s(x') = b$, and $b_s(y) = \beta$. Then, the continuity of v^s implies that for small enough $\epsilon > 0$

$$\begin{aligned} \Pi^s(b_s(x_d)|x_d) &= \mathbf{E}\left[\left(v^s(x_d, x_d, \tilde{Y}_k) - b_s(x_d)\right) 1_{\{\tilde{Y}_k \leq x_d\}} \middle| \tilde{X}_1 = x_d\right] \\ &< \mathbf{E}\left[\left(v^s(x_d - \epsilon, x_d, \tilde{Y}_k) - b_s(x_d - \epsilon)\right) 1_{\{\tilde{Y}_k \leq x_d - \epsilon\}} \middle| \tilde{X}_1 = x_d\right] \\ &= \Pi(b_s(x_d - \epsilon)|x_d), \end{aligned}$$

which contradicts our assumption that $b_s(x_d)$ is an equilibrium bid.

The possibility that $\lim_{x \downarrow x_d} b_s(x) > b_s(x_d)$ can be ruled out similarly. ∎

We record a technical lemma before proving Proposition 3.

Lemma 6 (Milgrom and Weber (1982a)) *Let $\rho(z)$ and $\sigma(z)$ be differentiable functions for which (i) $\rho(\underline{x}) \geq \sigma(\underline{x})$ and (ii) $\rho(z) < \sigma(z)$ implies $\rho'(z) \geq \sigma'(z)$. Then $\rho(z) \geq \sigma(z)$ for all $z \geq \underline{x}$.*

PROOF OF PROPOSITION 3: Let $x_0 \geq x_0'$. By affiliation we know

$$\bar{b}(\underline{x}; x_0) = w^d(\underline{x}, \underline{x}, \underline{x}; x_0) \geq \bar{b}(\underline{x}; x_0') = w^d(\underline{x}, \underline{x}, \underline{x}; x_0').$$

If we can show that $\bar{b}(x; x_0) < \bar{b}(x; x_0')$ implies $\bar{b}'(x; x_0) \geq \bar{b}'(x; x_0')$, then the proposition follows from Lemma 6. Suppose that $\bar{b}(x; x_0) < \bar{b}(x; x_0')$. As generalizations of Lemmas 1, 2, and 3, we have that $w_1^d(x, x', y; x_0)$ is increasing in both y and x_0, $F_k(y|x; x_0)/f_k(y|x; x_0)$ is decreasing in x_0, and that $F_k(\cdot|x; x_0)/F_k(\cdot|x; x_0)$ dominates $F_k(\cdot|x; x_0')/F_k(\cdot|x; x_0')$ in the sense of first degree stochastic dominance. Then

$$\begin{aligned} \bar{b}'(x; x_0) &= (w^d(x, x, x; x_0) - \bar{b}(x; x_0))\frac{f_k(x|x; x_0)}{F_k(x|x; x_0)} + \int_{\underline{x}}^{x} w_1^d(x, x, y; x_0)\frac{f_k(y|x; x_0)}{F_k(x|x; x_0)}dy \\ &\geq (w^d(x, x, x; x_0') - \bar{b}(x; x_0'))\frac{f_k(x|x; x_0')}{F_k(x|x; x_0')} + \int_{\underline{x}}^{x} w_1^d(x, x, y; x_0')\frac{f_k(y|x; x_0')}{F_k(x|x; x_0)}dy \\ &= \bar{b}'(x; x_0'), \end{aligned}$$

which was to be shown. ∎

PROOF OF THEOREM 3: PROOF. Given Proposition 3, the proof of theorem 3 mimics that of Milgrom and Weber (1982a, Theorem 16).

Define

$$W(x,z) \equiv \mathbf{E}\left[\bar{b}(x;\tilde{X}_0)|\tilde{Y}_k \leq x, \tilde{X}_1 = z\right],$$

which is the expected price paid by bidder 1 when the auctioneer publicly reveals $\tilde{X}_0$, conditional on bidder 1 winning when $\tilde{X}_1 = z$ and bidder 1 bids as if $\tilde{X}_1 = x$. By Proposition 3 and by the hypothesis that $\tilde{X}_0$ and $\tilde{X}_1$ are affiliated, $W_2(x,z) \geq 0$. Note that, by symmetry, the expected revenue for the seller under the policy of publicly reporting $\tilde{X}_0$ is k times

$$\begin{aligned}
&\mathbf{E}\left[\bar{b}(\tilde{X}_1;\tilde{X}_0)|\{\tilde{Y}_k \leq \tilde{X}_1\}\right] \\
= \;&\mathbf{E}\left[\mathbf{E}\left[\bar{b}(\tilde{X}_1;\tilde{X}_0)|\tilde{Y}_k \leq \tilde{X}_1, \tilde{X}_1\right]|\{\tilde{Y}_k \leq \tilde{X}_1\}\right] \\
= \;&\mathbf{E}\left[W(\tilde{X}_1,\tilde{X}_1)|\{\tilde{Y}_k \leq \tilde{X}_1\}\right],
\end{aligned}$$

where the first equality follows from the law of iterative expectations and the second from the definition of W. On the other hand, without reporting $\tilde{X}_0$, the expected revenue for the seller is k times

$$\mathbf{E}\left[\bar{b}(\tilde{X}_1)|\{\tilde{Y}_k \leq \tilde{X}_1\}\right].$$

If we can show that $W(x,x) \geq \bar{b}(x)$ then we are done. We will utilize Milgrom and Weber [1982, Lemma 2]. Note first that by the law of iterative expectations,

$$\begin{aligned}
W(\underline{x},\underline{x}) &= \mathbf{E}\left[\bar{b}(\underline{x};\tilde{X}_0)|\tilde{Y}_k \leq \underline{x}, \tilde{X}_1 = \underline{x}\right] \\
&= \mathbf{E}\left[w(\underline{x},\underline{x},\underline{x};\tilde{X}_0)|\tilde{Y}_k = \underline{x}, \tilde{X}_1 = \underline{x}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\tilde{V}|\tilde{X}_0, \tilde{X}_1 = \underline{x}, \tilde{Y}_k = \underline{x}\right]|\tilde{Y}_k \leq \underline{x}, \tilde{X}_1 = \underline{x}\right] \\
&= v(\underline{x},\underline{x},\underline{x}) = \bar{b}(\underline{x}).
\end{aligned}$$

Now we claim that $W(x,x) < \bar{b}(x)$ implies $dW(x,x)/dx \geq \bar{b}'(x)$. Note first that if bidder 1, prior to learning $\tilde{X}_0$ but after observing $\tilde{X}_1 = x$, were to commit himself to some bidding strategy $\bar{b}(z;\cdot)$, his optimal choice will be $z = x$, since $\bar{b}(x;x_0)$ is optimal when $\tilde{X}_0 = x_0$. Thus $\hat{W}(z,x)$ at $z = x$ will have to satisfy the first order condition

$$W_1(x,x) = (v(x,x,x) - W(x,x))\frac{f_k(x|x)}{F_k(x|x)} + \int_{\underline{x}}^{x} v_1(x,x,y)\frac{f_k(y|x)}{F_k(x|x)}dy. \tag{25}$$

Then

$$\begin{aligned}\bar{b}'(x) &= (v(x,x,x)-\bar{b}(x))\frac{f_k(x|x)}{F_k(x|x)}+\int_{\underline{x}}^{x} v_1(x,x,y)\frac{f_k(y|x)}{F_k(x|x)}dy\\ &\le (v(x,x,x)-W(x,x))\frac{f_k(x|x)}{F_k(x|x)}+\int_{\underline{x}}^{x} v_1(x,x,y)\frac{f_k(y|x)}{F_k(x|x)}dy\\ &\le W_1(x,x)+W_2(x,x)=\frac{dW(x,x)}{dx},\end{aligned}$$

where the first equality follows from (6), the first inequality follows from the hypothesis that $W(x,x)<\hat{b}(x)$, and the second inequality follows from (25) and the fact that $W_2(x,x)\ge 0$. The assertion then follows from Milgrom and Weber [1982, Lemma 2]. ∎

PROOF OF THEOREM 4: Let $P^u(x)$ denote the expected price paid by bidder 1 in a uniform-price auction, conditional upon winning, when bidders use strategy b_0 and $\tilde{X}_1=x$. That is

$$\begin{aligned}P^u(x) &\equiv \int_{\underline{x}}^{x} b_0(y)\frac{f_k(y|x)}{F_k(x|x)}dy\\ &= \int_{\underline{x}}^{x}(v^u(y,y,y)+\frac{h(y)}{f_k(y|y)})\frac{f_k(y|x)}{F_k(x|x)}dy. \qquad (26)\end{aligned}$$

If $P^d(x)$ denotes the corresponding expected price when bidders use strategy $\hat{b}$ in a discriminatory auction then

$$\begin{aligned}P^d(x) &\equiv \hat{b}(x) \qquad (27)\\ &= \int_{\underline{x}}^{x}(v^d(y,y,y)+\frac{h(y)}{f_k(y|y)})dL(y|x)\\ &= \int_{\underline{x}}^{x} b_0(y)dL(y|x) \qquad (28)\end{aligned}$$

where the first equality follows from using integration by parts on (7) and the second from (14). Note that $L(u|x)$ and $\frac{F_k(y|x)}{F_k(x|x)}$ are probability distributions on $[\underline{x},x]$. Then since $b_0(y)$ is increasing, the proof is complete if we can show that $\frac{F_k(y|x)}{F_k(x|x)}$ stochastically dominates $L(u|x)$ in the sense of first-order, that is, $\frac{F_k(y|x)}{F_k(x|x)}\le L(u|x)$, $\forall y$.

Since

$$\ln(F_k(y|x))=\int_{\underline{x}}^{y}\frac{f_k(s|x)}{F_k(s|x)}ds$$

we have

$$F_k(y|x)=\exp\{\int_{\underline{x}}^{y}\frac{f_k(s|x)}{F_k(s|x)}ds\}.$$

Therefore

$$\begin{aligned}\frac{F_k(y|x)}{F_k(x|x)} &= \exp\{-\int_y^x \frac{f_k(s|x)}{F_k(s|x)}ds\} \\ &\leq \exp\{-\int_y^x \frac{f_k(s|s)}{F_k(s|s)}ds\} \\ &= L(y|x)\end{aligned}$$

where the inequality follows from Lemma 2. ▮

PROOF OF PROPOSITION 4: Let $(b_1, b_2, \ldots, b_n)$ be a candidate for Nash equilibrium where $b_i : [\underline{x}, \overline{x}] \mapsto \Re$ are strictly increasing. We will show that if bidder j, uses strategy b_j, $j = 2, 3, \ldots, n$ and the resale market buyers believe that each bidder i uses strategy b_i, $i = 1, 2, \ldots, n$ then bidder 1 has an incentive to deviate from $b_1(\tilde{X}_1)$. In fact we will show that when $\tilde{X}_1 = x$, bidder $1's$ profits are *minimized* at a bid of $b_1(x)$.

The price that bidder 1 faces is

$$\tilde{B}_1 \equiv \max\{b_2(\tilde{X}_2), b_3(\tilde{X}_3), \ldots, b_n(\tilde{X}_n)\}.$$

Since b_i are strictly increasing, $\tilde{X}_1$ and $\tilde{B}_1$ are strictly affiliated and $\tilde{B}_1$ is atomless. We assume, for simplicity, that $\tilde{B}_1$ has a density function. The expected resale price if bidder 1 wins with a bid equal to b is[15]

$$r^u(b_1^{-1}(b), \tilde{B}_1) \equiv \mathbf{E}\left[\tilde{V}\middle|\tilde{X}_1 = b_1^{-1}(b), \tilde{B}_1\right]$$

The expected profit for bidder 1 if $\tilde{X}_1 = x$ and he bids $b_1(x')$ is

$$\begin{aligned}\Pi^u(x'|x) &\equiv \mathbf{E}\left[(r^u(x', \tilde{B}_1) - B_1))1_{\{b_1(x') \geq \tilde{B}_1\}}\middle|\tilde{X}_1 = x\right] \\ &= \int_{\underline{b}}^{b_1(x')} \left(r^u(x', \beta) - \beta\right) g(\beta|x) d\beta,\end{aligned}$$

where $\underline{b} \equiv \min\{b_2(\underline{x}), b_3(\underline{x}), \ldots, b_n(\underline{x})\}$, and $g(\cdot|x)$ is the conditional density of $\tilde{B}_1$ given $\tilde{X}_1 = x$. The first-order necessary condition for b_1 to be an equilibrium strategy is

$$\frac{\partial \Pi^u(x'|x)}{\partial x'}\Big|_{x'=x} = \left(r^u(x, b_1(x)) - b_1(x)\right) g(b_1(x)|x) + \int_{\underline{b}}^{b_1(x)} r_1^u(x, \beta) g(\beta|x) d\beta = 0,$$

where $r_1^u(x, \beta)$ denotes the derivative of r^u with respect to its first argument.

---

[15] Here we assume that the identity of the winning bidder is also disclosed. In our earlier analysis, since we restricted attention to symmetric equilibria, such an assumption was not necessary.

Let $x' > x$. Then by strict affiliation

$$
\begin{aligned}
0 &= (r^u(x', b_1(x')) - b_1(x'))\, g(b_1(x')|x') + \int_{\underline{b}}^{b_1(x')} r_1^u(x', \beta) g(\beta|x') d\beta \\
&< g(b_1(x')|x') \left((r^u(x', b_1(x')) - b_1(x')) + \int_{\underline{b}}^{b_1(x')} r_1^u(x', \beta) \frac{g(\beta|x)}{g(b_1(x')|x)} d\beta \right) \\
&= \frac{g(b_1(x')|x')}{g(b_1(x')|x)} \frac{\partial \Pi(x'|x)}{\partial x'}.
\end{aligned}
$$

Similarly, we can show that, for $x' < x$,

$$\frac{\partial \Pi(x'|x)}{\partial x'} < 0.$$

Thus for any $x \in [\underline{x}, \overline{x}]$, $\Pi^u(x'|x)$ achieves a global minimum at $x' = x$! ∎

PROOF OF THEOREM 5: Given that bidders $2, 3, \ldots, n$ use b^*, we can rewrite (21) as

$$\hat{\Pi}^u(b|x) = \int_{\underline{x}}^{b^{*-1}(b)} (\hat{v}^u(x, y) - \hat{v}^u(y, y)) f_k(y|x) dy, \tag{29}$$

where $f_k(y|x)$ is the conditional density of $\tilde{Y}_k$ given $\tilde{X}_1$. Since, by strict affiliation, $\hat{v}^u$ is strictly increasing in both arguments, the integrand in (29) is positive if and only if $x > y$. Thus bidder $1's$ profits are maximized when he wins if and only if $\{\tilde{X}_1 \geq \tilde{Y}_k\}$. Therefore bidder $1's$ profits are maximized[16] if he uses the strategy b^*. ∎

PROOF OF LEMMA 4: From the definition of b^* we have, if $\tilde{Y}_k \neq \overline{x}$,

$$
\begin{aligned}
b^*(\tilde{Y}_k) &\equiv \mathbf{E}[\mathbf{E}[\tilde{V} \big| \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k, \tilde{P}] \big| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k] \\
&> \mathbf{E}[\mathbf{E}[\tilde{V} \big| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k, \tilde{P}] \big| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k] \\
&= \mathbf{E}[\tilde{V} \big| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k],
\end{aligned}
$$

where the inequality follows from strict affiliation, and the equality from the law of iterative expectations. Note that the strict inequality above is an equality when $\tilde{Y}_k = \overline{x}$, which is a zero probability event. ∎

PROOF OF LEMMA 5: Let b^* be the equilibrium bidding strategy when $\tilde{P}$ is a strictly affiliated random variable, and let b_c^* be the equilibrium bidding strategy when $\tilde{P}$ is constant. Then since

$$\mathbf{E}[\mathbf{E}[\tilde{V} \big| \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k] \big| \tilde{X}_1, \tilde{Y}_k] = \mathbf{E}[\tilde{V} \big| \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k],$$

[16] If $\tilde{P}$ is independent of $\tilde{X}_i$, or if there is no post-auction public information, then $\hat{v}^u$ is constant in its first argument and no bid gives bidder i an expected profit greater than zero. However, $(b^*, \ldots, b^*)$ remains an equilibrium.

we have

$$\begin{aligned}
b_c^*(x) &\equiv \mathbf{E}[\mathbf{E}[\tilde{V}\big|\tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k]\big|\tilde{X}_1 = x, \tilde{Y}_k = x] \\
&= \mathbf{E}[\tilde{V}\big|\tilde{X}_1 \geq x, \tilde{Y}_k = x] \\
&= \mathbf{E}[\tilde{V}\big|\tilde{Z}_{k+1} = x],
\end{aligned} \tag{30}$$

where the last equality follows since the signals are identically distributed.

Next, when $x \neq \overline{x}$,

$$\begin{aligned}
b^*(x) &\equiv \mathbf{E}[\mathbf{E}[\tilde{V}\big|\tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k, \tilde{P}]\big|\tilde{X}_1 = x, \tilde{Y}_k = x] \\
&= \mathbf{E}[\mathbf{E}[\tilde{V}\big|\tilde{Z}_{k+1}, \tilde{P}]\big|\tilde{Z}_k = x, \tilde{Z}_{k+1} = x] \\
&< \mathbf{E}[\mathbf{E}[\tilde{V}\big|\tilde{Z}_{k+1}, \tilde{P}]\big|\tilde{Z}_k \geq x, \tilde{Z}_{k+1} = x] \\
&= \mathbf{E}[\mathbf{E}[\tilde{V}\big|\tilde{Z}_{k+1}, \tilde{P}]\big|\tilde{Z}_{k+1} = x] \\
&= \mathbf{E}[\tilde{V}\big|\tilde{Z}_{k+1} = x] \\
&= b_c^*(x),
\end{aligned}$$

where the first equality follows since the signals are identically distributed, the inequality from strict affiliation, the second equality from the definition of Z_k, and the last equality from (30). When $x = \overline{x}$, which is a zero probability event, the above strict inequality becomes an equality. Thus the assertion of lemma is proved. ∎

Proof of Theorem 6: By symmetry, the expected revenue for the auctioneer is equal to n times the unconditional expected payment of bidder 1. Let $\hat{R}^u$ and R^d denote the expected revenue of the auctioneer under uniform-price auction and under discriminatory auction, respectively. Then

$$\begin{aligned}
\hat{R}^u &= n \times \mathbf{E}\left[b^*(\tilde{Y}_k)1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}\right] \\
&= k \times \mathbf{E}\left[b^*(\tilde{Y}_k)\big|\tilde{X}_1 \geq \tilde{Y}_k\right],
\end{aligned}$$

$$\begin{aligned}
R^d &= n \times \mathbf{E}\left[\hat{b}(\tilde{X}_1)1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}\right] \\
&= k \times \mathbf{E}\left[\hat{b}(\tilde{X}_1)\big|\tilde{X}_1 \geq \tilde{Y}_k\right].
\end{aligned}$$

Note that the total unconditional expected profits for bidders in equilibrium for the uniform-price auction and for the discriminatory auction are, respectively,

$$\begin{aligned}
& n \times \mathbf{E}\left[(\hat{r}^u(\tilde{Y}_k, \tilde{P}) - b^*(\tilde{Y}_k))1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}\right] \\
= \ & k \times \mathbf{E}\left[\tilde{V}\big|\tilde{X}_1 \geq \tilde{Y}_k\right] - \hat{R}^u,
\end{aligned}$$

and

$$\begin{aligned} & n \times \mathbf{E}\left[(r^d(\tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{Y}_k, \tilde{P}) - b^*(\tilde{X}_1))1_{\{\tilde{X}_1 \geq \tilde{Y}_k\}}\right] \\ = \;& k \times \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] - R^d. \end{aligned}$$

Thus, before bidders receive their private information, the two auctions are constant-sum games between the auctioneer and the bidders, with total payoff equal to $k \times \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 \geq \tilde{Y}_k\right]$. Putting

$$\hat{r}^u(\tilde{Y}_k) \equiv \mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k\right],$$

it is easily seen that

$$\mathbf{E}\left[\hat{r}^u(\tilde{Y}_k, \tilde{P}) - \hat{r}^u(\tilde{Y}_k) \middle| \tilde{X}_1 \geq \tilde{Y}_k, \tilde{Y}_k\right] = 0,$$

and hence

$$\mathbf{E}\left[\hat{r}^u(\tilde{Y}_k, \tilde{P}) - \hat{r}^u(\tilde{Y}_k) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] = 0.$$

Thus

$$R^o \equiv k \times \mathbf{E}\left[\hat{r}^u(\tilde{Y}_k) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] > R^d = k \times \mathbf{E}\left[\hat{b}(\tilde{X}_1) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right],$$

since the (unconditional) expected profit of a bidder in a discriminatory auction is always strictly positive by strict affiliation.

For ease of exposition, we assume that the support of $\tilde{P}$ is finite. Then $M \equiv (R^o - R^d)/(k(\overline{p} - \underline{p}))$ is strictly positive. We will show that if $\partial \hat{r}^u(y,p)/\partial p \leq M$ for all $y, p \in [\underline{x}, \overline{x}] \times [\underline{p}, \overline{p}]$, then $\hat{R}^u \geq R^d$. First note that by affiliation $\partial \hat{r}^u(y,p)/\partial p \geq 0$. Thus

$$\begin{aligned} \hat{r}^u(\tilde{Y}_k, \tilde{P}) &\geq \hat{r}^u(\tilde{Y}_k, \overline{p}) - M(\overline{p} - \tilde{P}) \\ &> \hat{r}^u(\tilde{Y}_k) - M(\overline{p} - \underline{p}), \end{aligned}$$

where the strict inequality follows from the assumption of strict affiliation. Hence

$$\hat{r}^u(\tilde{Y}_k) - \hat{r}^u(\tilde{Y}_k, \tilde{P}) < (R^o - R^d)/k,$$

and thus

$$\begin{aligned} -b^*(\tilde{Y}_k) &= -\mathbf{E}\left[\hat{r}^u(\tilde{Y}_k, \tilde{P}) \middle| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k\right] \\ &< (R^o - R^d)/k - \hat{r}^u(\tilde{Y}_k). \end{aligned}$$

Taking expectation of the above expression conditional on $\{\tilde{X}_1 \geq \tilde{Y}_k\}$ gives

$$\hat{R}^u = k \times \mathbf{E}\left[b^*(\tilde{Y}_k) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] > R^d,$$

which was to be shown. ∎

PROOF OF THEOREM 7: From Milgrom and Weber (1982a, Theorem 15) and the hypothesis that $(\tilde{V}, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$ are strictly affiliated, it follows that

$$D \equiv k \times \mathbf{E}\left[\mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k\right] \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] - k \times \mathbf{E}\left[\mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1, \tilde{Y}_k = \tilde{X}_1\right] - J(\tilde{X}_1) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] > 0,$$

where

$$J(x) \equiv \int_{\underline{x}}^{x} L(u|x) dt(u),$$

and where $t(u)$ and $L(u|x)$ are as defined in (8).[17]

Let $M \equiv D/(k\mathbf{E}[\tilde{X}_1 | \tilde{X}_1 \geq \tilde{Y}_k])$. We now show that with M as defined, the theorem is true. First we recall from Lemma 4 that, with probability one,

$$b^*(\tilde{Y}_k) > \mathbf{E}[\tilde{V} | \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k].$$

Therefore,

$$\mathbf{E}[b^*(\tilde{Y}_k) | \tilde{X}_1 \geq \tilde{Y}_k] > \mathbf{E}[\mathbf{E}[\tilde{V} | \tilde{X}_1 = \tilde{Y}_k, \tilde{Y}_k] | \tilde{X}_1 \geq \tilde{Y}_k]$$

and

$$k \times \mathbf{E}[b^*(\tilde{Y}_k) | \tilde{X}_1 \geq \tilde{Y}_k] - k \times \mathbf{E}[\mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1, \tilde{Y}_k = \tilde{X}_1\right] - J(\tilde{X}_1) | \tilde{X}_1 \geq \tilde{Y}_k] > D. \tag{31}$$

Next note that

$$\mathbf{E}\left[\tilde{V} \middle| \tilde{X}_1, \tilde{Y}_k = \tilde{X}_1\right] - J(\tilde{X}_1) = \hat{b}(\tilde{X}_1) - K(\tilde{X}_1),$$

where $\hat{b}$ is defined in (7) and

$$K(x) \equiv \int_{\underline{x}}^{x} \frac{h(u)}{f_k(u|u)} dL(u|x),$$

and where $h(u)$ and $L(u|x)$ are as defined in (8). Next the hypothesis that $\partial r^d(x, y_1, \ldots, y_k, p)/\partial x \leq M$ for all $x, y_1, \ldots, y_k, p$ implies that

$$k \times K(x) \leq M \times k \times x.$$

Hence

$$k \times \mathbf{E}\left[K(\tilde{X}_1) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] \leq D. \tag{32}$$

Substituting (32) into (31) gives

$$k \times \mathbf{E}\left[b^*(\tilde{Y}_k) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right] > k \times \mathbf{E}\left[\hat{b}(\tilde{X}_1) \middle| \tilde{X}_1 \geq \tilde{Y}_k\right],$$

which was to be shown. ∎

[17] Note that to prove this we also need a technical lemma that is slightly different from Lemma 6: Let $\rho(z)$ and $\sigma(z)$ be differentiable functions for which (i) $\rho(\underline{z}) \geq \sigma(\underline{z})$ and (ii) $\rho(z) \leq \sigma(z)$ implies $\rho'(z) > \sigma'(z)$. Then $\rho(z) > \sigma(z)$ for all $z > \underline{z}$. The reader should convince herself/himself that this is indeed true.

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